

# 2-permutations of lattice vertex operator algebras: Higher rank

Chongying Dong\*

Department of Mathematics, University of California, Santa Cruz, CA 95064  
USA

Feng Xu<sup>†</sup> and Nina Yu

Department of Mathematics, University of California, Riverside, CA 92521  
USA

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## Abstract

The fusion rules of the 2-permutation orbifold of an arbitrary lattice vertex operator algebra are determined by using the theory of quantum dimension.

## 1 Introduction

This paper is a continuation of our investigation on 2-permutation of lattice vertex operator algebras [DXY]. In particular, the quantum dimensions of irreducible modules and the fusion rules are determined. If the rank of the lattice is one, these results have been obtained previously in [DXY].

Let  $V$  be a vertex operator algebra and  $n$  a fixed positive integer and consider the tensor product vertex operator algebra  $V^{\otimes n}$  [FHL]. Then the symmetric group  $S_n$  acts naturally on  $V^{\otimes n}$  as automorphisms. The permutation orbifold theory has been studied extensively in physics [KS, FKS, BHS, Ba]. Conformal nets approach to permutation orbifolds have been given in [KLX]. Twisted sectors of permutation orbifolds of tensor products of an arbitrary vertex operator algebra have been constructed in [BDM]. The  $C_2$ -cofiniteness of permutation orbifolds and general cyclic orbifolds have been studied in [A3, A4, M]. But the representation theory such as rationality, classification of irreducible modules, and fusion rules for the

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fixed point vertex operator algebra  $(V^{\otimes n})^G$  for any  $n$  and any subgroup  $G$  of  $S_n$  have not been investigated much.

As a starting point, we studied representations of 2-permutation orbifold model of rank one lattice vertex operator algebras in [DXY]. In this paper, we complete the study of 2-permutation orbifold model of lattice vertex operator algebras  $V_L$  for any positive definite even lattice  $L$ . Similar to rank one case, the permutation orbifold model  $(V_L \otimes V_L)^{\mathbb{Z}_2}$  can be realized as a simple current extension of the rational vertex operator algebra  $V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}^+$ . It follows from [Y, HKL] that  $(V_L \otimes V_L)^{\mathbb{Z}_2}$  is rational. According to [DRX], every irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -module occurs in an irreducible  $g$ -twisted  $V_L \otimes V_L$ -module. So the classification of irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules is known. But this classification result does not suggest how to compute the fusion rules among the irreducible modules. The main idea is to use the general theory of simple current extension of a rational vertex operator algebra and representations of  $V_L$  and  $V_L^+$  to study the representations of  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ . We decompose each irreducible  $V_L \otimes V_L$ -module into a direct sum of irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules by using the fusion rules for both vertex operator algebras  $V_{\sqrt{2}L}$  and  $V_{\sqrt{2}L}^+$  [DL1, A1, ADL]. This decomposition is crucial in computing the fusion rules. We emphasize that the theory of quantum dimensions introduced and studied in [DJX, DRX] plays an essential role in computing the fusion rules. It is not clear to us how to achieve this without using the quantum dimensions. The fusion rules in conformal nets for any 2-permutation models were computed by using the  $S$ -matrix [KLX].

We should mention that the constructions of  $g$ -twisted modules for lattice vertex operator algebra  $V_L$  where  $g$  is automorphism of finite order induced from an isometry of  $L$  were already given in [FLM1, FLM2, L, DL2]. In the case  $g$  is of order 2, the irreducible modules of  $V_L^{(g)}$  have been classified recently in [BE]. An equivalence of two constructions of permutation-twisted modules for lattice vertex operator algebras in [FLM1, L] and [BDM] was given in [BHL].

The paper is organized as follows: §2 and §3 are preliminaries on the vertex operator algebras theory. In these sections we give some basic notions that appear in this paper and recall the constructions of the lattice type vertex operator algebras  $V_L$  and  $V_L^+$  and their (twisted) modules. In §4 we study  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ , the 2-cyclic permutation orbifold models for rank  $d$  lattice vertex operator algebras. In particular, we decompose each irreducible  $V_L \otimes V_L$ -module into a direct sum of irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules. The quantum dimensions of all irreducible modules of  $(V_L \otimes V_L)^{\mathbb{Z}_2}$  are obtained explicitly in §5. Finally, we apply results from the previous sections to determine all fusion products in §6.

## 2 Preliminaries

Let  $(V, Y, \mathbf{1}, \omega)$  be a vertex operator algebra [Bo, FLM2] and  $g$  an automorphism of vertex operator algebra  $V$  of order  $T$ . Denote the decomposition of  $V$  into eigenspaces of  $g$  as:

$$V = \bigoplus_{r=0}^{T-1} V^r, \quad V^r = \{v \in V \mid gv = e^{2\pi ir/T} v\}.$$

Here are the definitions of weak, admissible, ordinary  $g$ -twisted  $V$ -modules [DLM3].

**Definition 2.1.** A weak  $g$ -twisted  $V$ -module  $M$  is a vector space with a linear map

$$Y_M : V \rightarrow (\text{End} M) \{z\}$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \quad (v_n \in \text{End} M)$$

which satisfies the following: for all  $0 \leq r \leq T-1$ ,  $u \in V^r$ ,  $v \in V$ ,  $w \in M$ ,

- (1)  $u_l w = 0$  if  $l$  is sufficiently large,
- (2)  $Y_M(u, z) = \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1}$ ,
- (3)  $Y_M(1, z) = Id_M$ ,
- (4) (Twisted Jacobi identity)

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2) Y_M(u, z_1)$$

$$z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2),$$

where  $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ .

**Definition 2.2.** An *admissible  $g$ -twisted  $V$ -module*  $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$  is a  $\frac{1}{T}\mathbb{Z}_+$ -graded weak  $g$ -twisted module such that  $u_m M(n) \subset M(\text{wt} u - m - 1 + n)$  for homogeneous  $u \in V$  and  $m, n \in \frac{1}{T}\mathbb{Z}$ .

**Definition 2.3.** A (ordinary)  *$g$ -twisted  $V$ -module* is a weak  $g$ -twisted  $V$ -module  $M$  which carries a  $\mathbb{C}$ -grading induced by the spectrum of  $L(0)$ , where  $L(0)$  is the component operator of  $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ . That is, we have  $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ , where  $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$ . Moreover,  $\dim M_\lambda$  is finite and for fixed  $\lambda$ ,  $M_{\frac{n}{T} + \lambda} = 0$  for all small enough integers  $n$ . A vector  $w \in M_\lambda$  is called a weight vector of weight  $\lambda$ , and write  $\lambda = \text{wt} w$ .

**Remark 2.4.** If  $g = Id_V$  we have the notions of weak, ordinary and admissible  $V$ -modules.

Note that the cyclic group  $\langle g \rangle$  generated by  $g$  acts on any admissible  $g$ -twisted  $V$ -module  $M$  such that  $g|_{M(n)} = e^{-2\pi i n}$  for  $n \in \frac{1}{T}\mathbb{Z}$  and  $gY_M(v, z)g^{-1} = Y_M(gv, z)$  for all  $v \in V$ . In particular,  $M^r = \bigoplus_{n \in \mathbb{Z}} M(\frac{r}{T} + n)$  is an admissible  $V^{(g)}$ -module for  $r = 0, \dots, T-1$ . Moreover, if  $M$  is irreducible then each  $M^r$  is irreducible admissible  $V^{(g)}$ -module [DY, MT, DRX].

**Definition 2.5.** A vertex operator algebra  $V$  is called  *$g$ -rational* if the admissible  $g$ -twisted module category is semisimple.  $V$  is called *rational* if  $V$  is 1-rational.

**Definition 2.6.** A vertex operator algebra  $V$  is said to be  *$C_2$ -cofinite* if  $V/C_2(V)$  is finite dimensional, where  $C_2(V) = \langle v_{-2}u \mid v, u \in V \rangle$ .

**Remark 2.7.** If vertex operator algebra  $V$  is rational or  $C_2$ -cofinite, then  $V$  has only finitely many irreducible admissible modules up to isomorphism and each irreducible admissible module is ordinary [DLM3, Li].

Now we consider the tensor product vertex algebras and the tensor product modules for tensor product vertex operator algebras. The tensor product of vertex operator algebras  $(V^1, Y^1, 1, \omega^1)$  and  $(V^2, Y^2, 1, \omega^2)$  is constructed on the tensor product vector space  $V = V^1 \otimes V^2$  where

the vertex operator  $Y(\cdot, z)$  is defined by  $Y(v^1 \otimes v^2, z) = Y(v^1, z) \otimes Y(v^2, z)$  for  $v^i \in V^i$ ,  $i = 1, 2$ , the vacuum vector is  $\mathbf{1} = 1 \otimes 1$  and the Virasoro element is  $\omega = \omega^1 \otimes \omega^2$ . Then  $(V, Y, \mathbf{1}, \omega)$  is a vertex operator algebra [FHL, LL]. Let  $W^i$  be an admissible  $V^i$ -module for  $i = 1, 2$ . We may construct the tensor product admissible module  $W^1 \otimes W^2$  for the tensor product vertex operator algebra  $V^1 \otimes V^2$  by  $Y(v^1 \otimes v^2, z) = Y(v^1, z) \otimes Y(v^2, z)$ . Then  $W^1 \otimes W^2$  is an admissible  $V^1 \otimes V^2$ -module. We have the following result about tensor product modules [DMZ, FHL]:

**Theorem 2.8.** Let  $V^1, V^2$  be rational vertex operator algebras, then  $V^1 \otimes V^2$  is rational and any irreducible  $V^1 \otimes V^2$ -module is a tensor product  $W^1 \otimes W^2$  for some irreducible  $V^i$ -module  $W^i$  and  $i = 1, 2$ .

Let  $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$  be an admissible  $g$ -twisted  $V$ -module, the contragredient module  $M'$  is defined as follows:

$$M' = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)^*,$$

where  $M(n)^* = \text{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$ . The vertex operator  $Y_{M'}(v, z)$  is defined for  $v \in V$  via

$$\langle Y_{M'}(v, z)f, u \rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})u \rangle$$

where  $\langle f, w \rangle = f(w)$  is the natural pairing  $M' \times M \rightarrow \mathbb{C}$ . Then  $M'$  is an admissible  $g^{-1}$ -twisted  $V$ -module [X]. A  $V$ -module  $M$  is said to be *self dual* if  $M$  and  $M'$  are isomorphic  $V$ -modules.

We now recall the notion of intertwining operators and fusion rules [FHL]:

**Definition 2.9.** Let  $(V, Y)$  be a vertex operator algebra and let  $(W^1, Y^1)$ ,  $(W^2, Y^2)$  and  $(W^3, Y^3)$  be  $V$ -modules. An intertwining operator of type  $\begin{pmatrix} W^1 \\ W^2 \ W^3 \end{pmatrix}$  is a linear map

$$I(\cdot, z) : W^2 \rightarrow \text{Hom}(W^3, W^1)\{z\}$$

$$u \mapsto I(u, z) = \sum_{n \in \mathbb{Q}} u_n z^{-n-1}$$

satisfying:

- (1) for any  $u \in W^2$  and  $v \in W^3$ ,  $u_n v = 0$  for  $n$  sufficiently large;
- (2)  $I(L_{-1}v, z) = (\frac{d}{dz})I(v, z)$ ;
- (3) (Jacobi Identity) for any  $u \in V$ ,  $v \in W^2$

$$\begin{aligned} z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^1(u, z_1) I(v, z_2) - z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) I(v, z_2) Y^3(u, z_1) \\ = z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right) I(Y^2(u, z_0)v, z_2). \end{aligned}$$

We denote the space of all intertwining operators of type  $\begin{pmatrix} W^1 \\ W^2 \ W^3 \end{pmatrix}$  by  $I_V \left( \begin{pmatrix} W^1 \\ W^2 \ W^3 \end{pmatrix} \right)$ .

Let  $N_{W^2, W^3}^{W^1} = \dim I_V \left( \begin{pmatrix} W^1 \\ W^2 \ W^3 \end{pmatrix} \right)$ . These integers  $N_{W^2, W^3}^{W^1}$  are usually called the *fusion rules*.

**Definition 2.10.** Let  $V$  be a vertex operator algebra, and  $W^1, W^2$  be two  $V$ -modules. A module  $(W, I)$ , where  $I \in I_V \left( \begin{smallmatrix} W \\ W^1 & W^2 \end{smallmatrix} \right)$ , is called a *fusion product* of  $W^1$  and  $W^2$  if for any  $V$ -module  $M$  and  $\mathcal{Y} \in I_V \left( \begin{smallmatrix} M \\ W^1 & W^2 \end{smallmatrix} \right)$ , there is a unique  $V$ -module homomorphism  $f : W \rightarrow M$ , such that  $\mathcal{Y} = f \circ I$ . As usual, we denote  $(W, I)$  by  $W^1 \boxtimes_V W^2$ .

It is well known that if  $V$  is rational, then the fusion product exists. We shall often consider the fusion product

$$W^1 \boxtimes_V W^2 = \sum_W N_{W^1, W^2}^W W$$

where  $W$  runs over the set of equivalence classes of irreducible  $V$ -modules.

The fusion rules satisfy the following symmetry [FHL].

**Proposition 2.11.** Let  $W^i$  ( $i = 1, 2, 3$ ) be  $V$ -modules. Then

$$\dim I_V \left( \begin{smallmatrix} W^3 \\ W^1 W^2 \end{smallmatrix} \right) = \dim I_V \left( \begin{smallmatrix} W^3 \\ W^2 W^1 \end{smallmatrix} \right), \dim I_V \left( \begin{smallmatrix} W^3 \\ W^1 W^2 \end{smallmatrix} \right) = \dim I_V \left( \begin{smallmatrix} (W^2)' \\ W^1 (W^3)' \end{smallmatrix} \right).$$

**Definition 2.12.** Let  $V$  be a simple vertex operator algebra. A simple  $V$ -module  $M$  is called a *simple current* if for any irreducible  $V$ -module  $W$ ,  $W \boxtimes M$  exists and is also a simple  $V$ -module.

Let  $D$  be a finite abelian group and assume that we have a set of irreducible simple current  $V^0$ -modules  $\{V^\alpha | \alpha \in D\}$  indexed by  $D$ . The following definition is from [Y].

**Definition 2.13.** An extension  $V_D = \bigoplus_{\alpha \in D} V^\alpha$  of  $V^0$  is called a  *$D$ -graded simple current extension* if  $V_D$  carries a structure of a simple vertex operator algebra such that  $Y(u^\alpha, z)u^\beta \in V^{\alpha+\beta}((z))$  for any  $u^\alpha \in V^\alpha$  and  $u^\beta \in V^\beta$ .

### 3 Vertex Operator algebra $V_L$ and $V_L^+$

We first review the construction of the vertex operator algebra  $V_L$  associated with a positive definite even lattice  $L$  [Bo, FLM2].

We are working in the setting of [DL1, FLM2]. Let  $L$  be a positive definite even lattice with bilinear form  $\langle \cdot, \cdot \rangle$  and  $L^\circ$  its dual lattice in  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ . Let  $\{\lambda_0 = 0, \lambda_1, \lambda_2, \dots\}$  be a complete set of coset representatives of  $L$  in  $L^\circ$ . Then the lattice vertex operator algebra  $V_L$  is rational and  $V_{\lambda_i + L}$  are the irreducible  $V_L$ -modules [Bo, FLM2, D1, DLM2].

Now assume  $M$  is positive definite even lattice such that  $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$  for  $\alpha, \beta \in M$ . In this case,  $V_{M+\lambda} = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathbb{C}[\lambda + M]$  for any  $\lambda \in M^\circ$  where  $S(\cdot)$  is the symmetric algebra and  $\mathbb{C}[\lambda + M] = \sum_{\alpha \in M} \mathbb{C}e_{\lambda+\alpha}$  is the subspace of the group algebra  $\mathbb{C}[M^\circ]$  corresponds to  $\lambda + M$ . Then  $V_M$  has a canonical automorphism  $\theta$  of order 2 induced from  $-1$  isometry of  $M$ . In fact, we can define a linear map  $\theta$  from  $V_{\lambda+M}$  to  $V_{-\lambda+M}$  for any  $\lambda \in M^\circ$  such that  $\theta Y_{\lambda+M}(u, z)\theta^{-1} = Y_{-\lambda+M}(\theta u, z)$  for any  $u \in V_M$  where  $Y_{\lambda+M}$  defines a  $V_M$ -module

structure on  $V_{\lambda+M}$  [AD]. Clearly, if  $2\lambda \in M$ ,  $\theta$  is an endomorphism from  $V_{\lambda+M}$  to  $V_{\lambda+M}$ . For such  $\lambda$  we denote eigenspace of  $\theta$  with eigenvalue  $\pm 1$  in  $V_{\lambda+M}$  by  $V_{\lambda+M}^\pm$ .

We now turn our attention to the construction of  $\theta$ -twisted  $V_M$ -modules [FLM1, FLM2, L, DL2]. Note that  $M/2M$  is an abelian group of order  $2^d$  where  $d$  is the rank of  $M$ . Then  $M/2M$  has exactly  $2^d$  inequivalent irreducible modules  $T_\chi$  where  $\chi$  is irreducible character of  $M/2M$ . It was proved in [FLM2] that  $V_M^{T_\chi} = S\left(\mathfrak{h} \otimes \left(t^{-\frac{1}{2}}\right) \mathbb{C}[t^{-1}]\right) \otimes T_\chi$  is an irreducible  $\theta$ -twisted module. According to Remark 2.4,  $\theta$  acts on  $V_M^{T_\chi}$ . Again we denote eigenspace of  $\theta$  with eigenvalue  $\pm 1$  by  $V_M^{T_\chi, \pm}$ . Moreover,  $V_M$  is  $\theta$ -rational and  $\{V_M^{T_\chi} | \chi\}$  gives a complete list of inequivalent irreducible  $\theta$ -twisted  $V_M$ -modules [D2].

We have the following classification of the irreducible  $V_L^+$ -modules [AD, DN]:

**Theorem 3.1.** Let  $M$  be a positive definite even lattice and  $\{\lambda_i\}$  be a set of coset representatives of  $M$  in  $M^\circ$ . Then any irreducible  $V_M^+$ -module is isomorphic to one of the following:

$$V_{\lambda_i+M}(2\lambda_i \notin M), V_{\lambda_i+M}^\pm(2\lambda_i \in M), V_M^{T_\chi, \pm}.$$

Furthermore,  $V_{\lambda_i+M} \cong V_{\lambda_j+M}$  if and only if  $\lambda_i \pm \lambda_j \in M$ .

From now on, we fix a rank  $d$  lattice  $L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_d$  with positive definite symmetric non-degenerate bilinear form  $\langle \cdot, \cdot \rangle$ . Let  $M = \sqrt{2}L$ . Later we will see that the 2-permutation orbifold model we study is closely related to the rational vertex operator algebras  $V_{\sqrt{2}L}$  and  $V_{\sqrt{2}L}^+$ . We now consider the fusion rules for the vertex operator algebras  $V_{\sqrt{2}L}$  and  $V_{\sqrt{2}L}^+$ .

First we notice that the dual lattice of  $\sqrt{2}L$  can be written by  $(\sqrt{2}L)^\circ = \left\{ \frac{\lambda}{\sqrt{2}} | \lambda \in L^\circ \right\}$ . Thus fusion rules for irreducible  $V_{\sqrt{2}L}$ -modules are given by the following [DL1]:

**Proposition 3.2.**  $N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L} \\ V_{\frac{\lambda}{\sqrt{2}} + \sqrt{2}L} V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L} \end{matrix} \right) = \delta_{\frac{\lambda+\mu}{\sqrt{2}} + \sqrt{2}L, \frac{\gamma}{\sqrt{2}} + \sqrt{2}L}$  for  $\lambda, \mu$  and  $\gamma \in L^\circ$ .

The fusion rules for  $V_L^+$  for any  $L$  was obtain in [ADL]. For this purpose, we need to identify the contragredient modules of the irreducible  $V_{\sqrt{2}L}^+$ -modules first (see Proposition 3.7 of [ADL]).

**Proposition 3.3.** Every irreducible  $V_{\sqrt{2}L}^+$ -module is self dual.

**Remark 3.4.** For any  $\lambda \in L^\circ$ , we define a character  $\chi_\lambda$  so that  $\chi_\lambda(\sqrt{2}\alpha_i) = (-1)^{\frac{\langle \alpha_i, \alpha_i \rangle}{2} + \langle \lambda, \alpha_i \rangle}$  for  $1 \leq i \leq d$ . Then  $\chi_\mu = \chi_\lambda$  if and only if  $\lambda - \mu \in 2L^\circ$ . Thus  $\{\chi_\lambda | \lambda \in L^\circ/2L^\circ\}$  gives all different characters.

Recall the number  $\pi_{\lambda, \mu} = e^{\langle \lambda, \mu \rangle \pi i}$  for  $\lambda, \mu \in (\sqrt{2}L)^\circ$  [ADL]. For any character  $\chi$  of  $\sqrt{2}L/2\sqrt{2}L$ ,  $c_\chi$  was defined in [ADL] and we note that here for any  $\alpha \in L$  we have:

$$c_\chi \left( \frac{\alpha}{\sqrt{2}} \right) = (-1)^{\langle \alpha, \alpha \rangle} \chi \left( \sqrt{2}\alpha \right) = \chi \left( \sqrt{2}\alpha \right).$$

For any  $\mu \in L^\circ, \alpha \in L$ , the character  $\chi_{\frac{\mu}{\sqrt{2}}}$  is defined in [ADL] by

$$\chi_{\frac{\mu}{\sqrt{2}}} \left( \sqrt{2}\alpha \right) = (-1)^{\langle \alpha, \mu \rangle} \chi_\lambda \left( \sqrt{2}\alpha \right)$$

and  $T_{\chi_\lambda \left(\frac{\mu}{\sqrt{2}}\right)}$  is denoted by  $T_{\chi_\lambda}^{\left(\frac{\mu}{\sqrt{2}}\right)}$ . By the definition of  $\chi_\lambda$  in Remark 3.4, it is easy to check that

for  $\alpha \in L$ ,  $\lambda \in L^\circ$ ,  $\chi_\lambda^{\left(\frac{\alpha}{\sqrt{2}}\right)}(\sqrt{2}\alpha_i) = \chi_{\lambda+\alpha}(\sqrt{2}\alpha_i)$  for  $1 \leq i \leq d$  and hence  $T_{\chi_\lambda}^{\left(\frac{\alpha}{\sqrt{2}}\right)} = T_{\chi_{\lambda+\alpha}}$  for any  $\lambda \in L^\circ$  and  $\alpha \in L$ .

A triple  $(\lambda, \mu, \gamma) \subset L^\circ$  is said to be an admissible triple modulo  $L$  if  $p\lambda + q\mu + r\gamma \in L$  for some  $p, q, r \in \{\pm 1\}$ . Now we are ready to list fusion rules of irreducible  $V_{\sqrt{2}L}^+$ -modules [ADL]:

**Proposition 3.5.** Let  $L$  be the rank  $d$  lattice as before. For any irreducible  $V_{\sqrt{2}L}^+$ -modules  $M^i$  ( $i = 1, 2, 3$ ), the fusion rule of type  $\left(\begin{smallmatrix} M^3 \\ M^1 M^2 \end{smallmatrix}\right)$  is 1 if and only if  $M^i$  ( $i = 1, 2, 3$ ) satisfy one of the following conditions:

- (i)  $M^1 = V_{\frac{\lambda}{\sqrt{2}} + \sqrt{2}L}$  for  $\lambda \in L^\circ$  such that  $\lambda \notin L$  and  $(M^2, M^3)$  is one of the following pairs:
  - $\left(V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}, V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}\right)$  for  $\mu, \gamma \in L^\circ$  such that  $\mu, \gamma \notin L$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,
  - $\left(V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}^\pm, V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}^\pm\right), \left(V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}, V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}^\pm\right)$  for  $\mu \in L, \gamma \in L^\circ$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,
  - $\left(V_{\sqrt{2}L}^{T_{\chi_\mu, \pm}}, V_{\sqrt{2}L}^{T_{\chi_\mu}^{\left(\frac{\lambda}{\sqrt{2}}\right), \pm}}\right), \left(V_{\sqrt{2}L}^{T_{\chi_\mu, \pm}}, V_{\sqrt{2}L}^{T_{\chi_\mu}^{\left(\frac{\lambda}{\sqrt{2}}\right), \mp}}\right)$  for any irreducible  $\sqrt{2}L/2\sqrt{2}L$ -module  $T_{\chi_\mu}$ .
- (ii)  $M^1 = V_{\frac{\lambda}{\sqrt{2}} + \sqrt{2}L}^+$  for  $\lambda \in L$  and  $(M^2, M^3)$  is one of the following pairs:
  - $\left(V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}, V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}\right)$  for  $\mu, \gamma \in L^\circ$  such that  $\mu \notin L$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,
  - $\left(V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}^\pm, V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}^\pm\right)$  for  $\mu \in L, \gamma \in L^\circ$  such that  $\pi_{\frac{\lambda}{\sqrt{2}}, \sqrt{2}\mu} = 1$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,
  - $\left(V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}^\pm, V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}^\mp\right)$  for  $\mu \in L, \gamma \in L^\circ$  such that  $\pi_{\frac{\lambda}{\sqrt{2}}, \sqrt{2}\mu} = -1$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,
  - $\left(V_{\sqrt{2}L}^{T_{\chi_\mu, \pm}}, V_{\sqrt{2}L}^{T_{\chi_{\mu+\lambda}, \pm}}\right), \left(V_{\sqrt{2}L}^{T_{\chi_{\mu+\lambda}, \pm}}, V_{\sqrt{2}L}^{T_{\chi_\mu, \pm}}\right)$  for any irreducible  $\sqrt{2}L/2\sqrt{2}L$ -module  $T_{\chi_\mu}$  such that  $\chi_\mu(\sqrt{2}\lambda) = 1$ ,
  - $\left(V_{\sqrt{2}L}^{T_{\chi_\mu, \pm}}, V_{\sqrt{2}L}^{T_{\chi_{\mu+\lambda}, \mp}}\right), \left(V_{\sqrt{2}L}^{T_{\chi_{\mu+\lambda}, \mp}}, V_{\sqrt{2}L}^{T_{\chi_\mu, \pm}}\right)$  for any irreducible  $\sqrt{2}L/2\sqrt{2}L$ -module  $T_{\chi_\mu}$  such that  $\chi_\mu(\sqrt{2}\lambda) = -1$ .
- (iii)  $M^1 = V_{\frac{\lambda}{\sqrt{2}} + \sqrt{2}L}^-$  for  $\lambda \in L$  and  $(M^2, M^3)$  is one of the following pairs:
  - $\left(V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}, V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}\right)$  for  $\mu, \gamma \in L^\circ$  such that  $\mu \notin L$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,
  - $\left(V_{\frac{\mu}{\sqrt{2}} + \sqrt{2}L}^\pm, V_{\frac{\gamma}{\sqrt{2}} + \sqrt{2}L}^\mp\right)$  for  $\mu \in L, \gamma \in L^\circ$  such that  $\pi_{\frac{\lambda}{\sqrt{2}}, \sqrt{2}\mu} = 1$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,



$\left(V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L}^{\pm}, V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L}^{\pm}\right)$  for  $\mu \in L, \gamma \in L^{\circ}$  such that  $\pi_{\frac{\lambda}{\sqrt{2}}, \sqrt{2}\mu} = -1$  and  $\left(\frac{\lambda}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$  is an admissible triple modulo  $\sqrt{2}L$ ,

$\left(V_{\sqrt{2}L}^{T_{\chi\mu}, \pm}, V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \mp}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \mp}, V_{\sqrt{2}L}^{T_{\chi\mu}, \pm}\right)$  for any irreducible  $\sqrt{2}L/2\sqrt{2}L$ -module  $T_{\chi\mu}$  such that  $\chi_{\mu}(\sqrt{2}\lambda) = 1$ ,

$\left(V_{\sqrt{2}L}^{T_{\chi\mu}, \pm}, V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \pm}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \pm}, V_{\sqrt{2}L}^{T_{\chi\mu}, \pm}\right)$  for any irreducible  $\sqrt{2}L/2\sqrt{2}L$ -module  $T_{\chi\mu}$  such that  $\chi_{\mu}(\sqrt{2}\lambda) = -1$ .

(iv)  $M^1 = V_{\sqrt{2}L}^{T_{\chi\mu}, +}$  for an irreducible  $\sqrt{2}L/2\sqrt{2}L$ -module  $T_{\chi\mu}$ , and  $(M^2, M^3)$  is one of the following pairs:

$$\begin{aligned} & \left(V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}, V_{\sqrt{2}L}^{T_{\chi\mu}^{(\frac{\lambda}{\sqrt{2}})}, \pm}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu}^{(\frac{\lambda}{\sqrt{2}})}, \pm}, V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}\right) \text{ for } \lambda \in L^{\circ} \text{ such that } \lambda \notin L, \\ & \left(V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}, V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \pm}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \pm}, V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}\right) \text{ for } \lambda \in L \text{ such that } \chi_{\mu}(\sqrt{2}\lambda) = 1, \\ & \left(V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}, V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \mp}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \mp}, V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}\right) \text{ for } \lambda \in L \text{ such that } \chi_{\mu}(\sqrt{2}\lambda) = -1. \end{aligned}$$

(v)  $M^1 = V_{\sqrt{2}L}^{T_{\chi\mu}, -}$  for an irreducible  $\sqrt{2}L/2\sqrt{2}L$ -module  $T_{\chi\mu}$ , and  $(M^2, M^3)$  is one of the following pairs:

$$\begin{aligned} & \left(V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}, V_{\sqrt{2}L}^{T_{\chi\mu}^{(\frac{\lambda}{\sqrt{2}})}, \pm}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu}^{(\frac{\lambda}{\sqrt{2}})}, \pm}, V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}\right) \text{ for } \lambda \in L^{\circ} \text{ such that } \lambda \notin L, \\ & \left(V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}, V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \mp}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \mp}, V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}\right) \text{ for } \lambda \in L \text{ such that } \chi_{\mu}(\sqrt{2}\lambda) = 1, \\ & \left(V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}, V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \pm}\right), \left(V_{\sqrt{2}L}^{T_{\chi\mu+\lambda}, \pm}, V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}\right) \text{ for } \lambda \in L \text{ such that } \chi_{\mu}(\sqrt{2}\lambda) = -1. \end{aligned}$$

## 4 The vertex operator algebra $(V_L \otimes V_L)^{\mathbb{Z}_2}$

Let  $L$  be the positive definite lattice as before. We consider the rational vertex operator algebra  $V_L \otimes V_L$  with the natural action of the 2-cycle  $\sigma = (1\ 2)$ . We denote the fixed point vertex operator subalgebra by  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ .

First we want to see  $(V_L \otimes V_L)^{\mathbb{Z}_2}$  is a simple current extension of vertex operator algebra  $V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}^+$ . For this purpose, we let  $L^+ = \{(x, x) \mid x \in L\}$ ,  $L^- = \{(x, -x) \mid x \in L\}$ . Then

$$\begin{aligned} L \oplus L &= \sum_{\alpha \in L} ((L^+ + L^-) + (\alpha, 0)) \\ &= \sum_{\alpha \in L} \left( L^+ + \frac{(\alpha, \alpha)}{2} \right) \oplus \left( L^- + \frac{(\alpha, -\alpha)}{2} \right) \end{aligned}$$

For any  $\alpha \in L$ , let  $\alpha^1 = (\alpha, \alpha)$ ,  $\alpha^2 = (\alpha, -\alpha)$ . Then  $\langle \alpha^1, \alpha^1 \rangle = \langle \alpha^2, \alpha^2 \rangle = 2 \langle \alpha, \alpha \rangle = \langle \sqrt{2}\alpha, \sqrt{2}\alpha \rangle$ . Note that  $\sigma$  also acts on  $L \oplus L$  so that  $\sigma(\alpha^1) = \alpha^1$  and  $\sigma(\alpha^2) = -\alpha^2$ . Let  $\mathcal{S}$  be



a set of coset representatives of  $2L$  in  $L$ . Then we have decomposition

$$V_{L \oplus L} = \sum_{\alpha \in S} V_{\frac{\alpha}{2} + L^+} \otimes V_{\frac{\alpha}{2} + L^-}.$$

It is clear that  $L^+ \cong L^- \cong \sqrt{2}L$ . Also,  $\sigma(\alpha^2) = \theta(\alpha^2) = -\alpha^2$  where  $\theta$  is the  $-1$ -isometry on  $L^-$  defined before. Thus

$$(V_{L \oplus L})^{\mathbb{Z}_2} \cong \sum_{\alpha \in S} V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^+.$$

For short, we set

$$\mathcal{U} = \sum_{\alpha \in S} V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^+$$

and

$$\mathcal{V} = V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}^+.$$

It follows from [DL1] and 3.5 that each  $V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^+$  is a simple current  $\mathcal{V}$ -module. In particular,  $\mathcal{U}$  is a simple current extension of  $\mathcal{V}$ . By [D1, DLM1, A2, DJL],  $\mathcal{V}$  is rational. We also know that  $\mathcal{V}$  is  $C_2$ -cofinite [ABD, Ya]. From [Y] or [HKL] we have

**Proposition 4.1.** The vertex operator algebra  $\mathcal{U}$  is rational.

From the classification of irreducible modules of  $\mathcal{V}$  [D1, DN, AD], every irreducible module has positive weight except the vertex operator algebra itself. A result from [DRX] gives:

**Proposition 4.2.** Every irreducible  $\mathcal{U}$ -module occurs in an irreducible  $\sigma^i$ -twisted  $V_L \otimes V_L$  module for  $i = 0, 1$ .

As far as representation theory concerns, it remains to compute the fusion rules for  $\mathcal{U}$ . But it is not so easy to achieve this goal with the irreducible modules given abstractly in [DRX]. On the other hand  $\mathcal{U}$  is a simple current extension of  $V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}^+$  and we know the fusion rules for both  $V_{\sqrt{2}L}$  and  $V_{\sqrt{2}L}^+$ , it is natural to use these results to determine the fusion rules for  $\mathcal{U}$ . In the rest of this section, we will realize each irreducible  $\mathcal{U}$ -module as a direct sum of irreducible  $\mathcal{V}$ -modules.

Recall from [D1] that all irreducible  $V_L$ -modules are given by  $V_{L+\lambda}$ ,  $\lambda \in L^\circ$ . Let  $\mathcal{T} = \{\lambda_0 = 0, \lambda_1, \lambda_2, \dots\}$  be a complete set of representatives of  $L$  in  $L^\circ$ . Assume that  $|\mathcal{T}| = |L^\circ/L| = l$ . Then there are exactly  $l$  inequivalent irreducible  $V_L$ -modules.

For any  $\lambda, \mu \in L^\circ$ ,  $V_{\lambda+L} \otimes V_{\mu+L}$  is an irreducible  $V_L \otimes V_L$ -module. If  $\lambda + L \neq \mu + L$ , then  $V_{\lambda+L} \otimes V_{\mu+L}$  and  $V_{\mu+L} \otimes V_{\lambda+L}$  are isomorphic irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules [DM, DY]. The number of such isomorphism classes of irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules is  $\frac{l^2-l}{2}$ .

When  $\lambda = \mu$ ,  $V_{\lambda+L} \otimes V_{\mu+L} + V_{\mu+L} \otimes V_{\lambda+L}$  split into two different representations of  $(V_L \otimes V_L)^{\mathbb{Z}_2}$  by [DY]. The number of such isomorphism classes of irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules is  $2l$ .

It is shown in [BDM] that there is one-to-one correspondence between the category of  $\sigma$ -twisted  $V_L \otimes V_L$ -modules and the category of  $V_L$ -modules. Thus the number of isomorphism classes of irreducible  $\sigma$ -twisted  $V_L \otimes V_L$ -module is also  $l$ . From [DY], each twisted module

can be decomposed into a direct sum of two irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules. The number of such isomorphism classes of irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules is  $2l$ .

Together, we have  $\frac{l^2+7l}{2}$  inequivalent irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules. The following result is immediate from [DRX].

**Proposition 4.3.** There are exactly  $\frac{l^2+7l}{2}$  inequivalent irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules.

We now realize each irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -module in terms of irreducible  $\mathcal{V}$ -modules.

**Proposition 4.4.** Let  $L = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_d$  be a rank  $d$  positive definite even lattice. Then any irreducible  $\mathcal{U}$ -module has the following form as an  $V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}^+$ -module:

(i) For  $\lambda, \mu \in L^\circ$  with  $\lambda + L \neq \mu + L$ ,

$$(\lambda\mu) = \sum_{\alpha \in \mathcal{S}} V_{\frac{\lambda+\mu+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda-\mu+\alpha}{\sqrt{2}}+\sqrt{2}L}.$$

(ii) For  $\lambda \in L^\circ$ ,

$$\widetilde{(\lambda 0)} = \sum_{\alpha \in \mathcal{S}} V_{\sqrt{2}\lambda + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^+,$$

$$\widetilde{(\lambda 1)} = \sum_{\alpha \in \mathcal{S}} V_{\sqrt{2}\lambda + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^-.$$

(iii) For  $\lambda \in L^\circ$ ,

$$\widehat{(\lambda 0)} = \sum_{\alpha \in \mathcal{S}, \chi_\lambda(\sqrt{2}\alpha)=1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\lambda+\alpha},+} + \sum_{\alpha \in \mathcal{S}, \chi_\lambda(\sqrt{2}\alpha)=-1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\lambda+\alpha},-},$$

$$\widehat{(\lambda 1)} = \sum_{\alpha \in \mathcal{S}, \chi_\lambda(\sqrt{2}\alpha)=1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\lambda+\alpha},-} + \sum_{\alpha \in \mathcal{S}, \chi_\lambda(\sqrt{2}\alpha)=-1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\lambda+\alpha},+}.$$

*Proof.* (i) Let  $\lambda, \mu \in L^\circ$ , with  $\lambda + L \neq \mu + L$ , then

$$\begin{aligned} & (\lambda + L) \oplus (\mu + L) \\ &= (\lambda, \mu) + \sum_{\alpha \in \mathcal{S}} \left( \frac{(\alpha, \alpha)}{2} + L^+ \right) \oplus \left( \frac{(\alpha, -\alpha)}{2} + L^- \right) \\ &= \sum_{\alpha \in \mathcal{S}} \left( \frac{\lambda + \mu}{\sqrt{2}} + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L \right) \oplus \left( \frac{\lambda - \mu}{\sqrt{2}} + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L \right) \end{aligned}$$

Notice that here  $2 \left( \frac{\lambda - \mu}{\sqrt{2}} \right) \notin \sqrt{2}L$ . Thus we obtain the following decomposition

$$V_{\lambda+L} \otimes V_{\mu+L} \cong \sum_{\alpha \in \mathcal{S}} V_{\frac{\lambda+\mu}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda-\mu}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}$$

as  $V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}^+$ -modules. The number of such pairs  $(\lambda, \mu)$  that give inequivalent irreducible  $\mathcal{U}$ -modules is  $\frac{l^2-l}{2}$ . We obtain  $\frac{l^2-l}{2}$  irreducible  $\mathcal{U}$ -modules in this way. Denote these modules by  $(\lambda, \mu)$ ,  $\lambda, \mu \in \mathcal{T}$ .

(ii) If  $\lambda = \mu$ , then  $V_{\frac{\lambda-\mu}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \cong V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^+ + V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^-$  as  $V_{\sqrt{2}L}^+$ -modules. Thus we have

$$\begin{aligned} V_{\lambda+L} \otimes V_{\lambda+L} &= \left( \sum_{\alpha \in \mathcal{S}} V_{\sqrt{2}\lambda+\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^+ \right) \\ &\quad + \left( \sum_{\alpha \in \mathcal{S}} V_{\sqrt{2}\lambda+\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^- \right) \end{aligned}$$

which is a sum of two irreducible  $\mathcal{U}$ -modules. Denote  $\sum_{\alpha \in \mathcal{S}} V_{\sqrt{2}\lambda+\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^+$  by  $(\widetilde{\lambda}, 0)$  and  $\sum_{\alpha \in \mathcal{S}} V_{\sqrt{2}\lambda+\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^-$  by  $(\widetilde{\lambda}, 1)$ . Then  $(\widetilde{\lambda}, \epsilon)$ ,  $\lambda \in \mathcal{T}$ ,  $\epsilon = 0, 1$  give all inequivalent irreducible  $\mathcal{U}$ -modules of this form. The number of such inequivalent irreducible  $\mathcal{U}$ -modules is  $2l$ .

(iii) The proof in this case is different from the cases (i), (ii). It is difficult to identify the irreducible  $\mathcal{U}$ -modules from decomposition of  $\sigma$ -twisted modules of  $V_L \otimes V_L$  directly. Note that  $V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}$  is a subalgebra of  $V_L \otimes V_L$  (see the discussion before Proposition 4.1). So any irreducible  $\sigma$ -twisted  $V_L \otimes V_L$ -module contains an irreducible  $1 \otimes \theta$ -twisted  $V_{\sqrt{2}L} \otimes V_{\sqrt{2}L}$ -module  $V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}^{T_{\chi\mu}}$  for some  $\lambda, \mu \in L^\circ$  [D2]. Moreover,  $V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}^{T_{\chi\mu}}$  is a direct sum of two irreducible  $\mathcal{V}$ -modules  $V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}^{T_{\chi\mu}, \pm}$ .

Since  $\mathcal{V}$  is a rational vertex operator subalgebra of  $\mathcal{U}$ , each irreducible  $\mathcal{U}$ -module  $M$  is a direct sum of irreducible  $\mathcal{V}$ -modules. For an irreducible  $\mathcal{V}$ -module  $W$ , we define  $\mathcal{U} \cdot W$  to be the fusion product of  $\mathcal{U}$  and  $W$  as  $\mathcal{V}$ -modules. That is,

$$\mathcal{U} \cdot W = \left( \sum_{\alpha \in \mathcal{S}} V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^+ \right) \boxtimes_{\mathcal{V}} W.$$

Since each  $V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^+$  is a simple current,  $\mathcal{U} \cdot W$  is a  $\mathcal{U}$ -module if and only if the weights of  $\mathcal{U} \cdot W$  lies in  $\mathbb{Z} + r$  for some  $r \in \mathbb{C}$ .

From the discussion above, we see that any irreducible  $\mathcal{U}$ -modules from the  $\sigma$ -twisted  $V_L \otimes V_L$ -modules has the form  $\mathcal{U} \cdot W$  where  $W = V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}^{T_{\chi\mu}, \pm}$  and  $\lambda, \mu \in L^\circ$ . First we take  $W = V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L}^{T_{\chi\mu}, +}$ . By fusion rules in Proposition 3.2 and Proposition 3.5, we get

$$\mathcal{U} \cdot W = \sum_{\alpha \in \mathcal{S}, \chi_\mu(\sqrt{2}\alpha)=1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L}^{T_{\chi\mu+\alpha}, +} + \sum_{\alpha \in \mathcal{S}, \chi_\mu(\sqrt{2}\alpha)=-1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L}^{T_{\chi\mu+\alpha}, -}.$$

Thus  $\mathcal{U} \cdot W$  is a  $\mathcal{U}$ -module only if  $\langle \lambda, \alpha \rangle + \frac{\langle \alpha, \alpha \rangle}{2} \in 2\mathbb{Z}$  for each  $\alpha \in \mathcal{S}$  that satisfies  $\chi_\mu(\sqrt{2}\alpha) = 1$ , and  $\langle \lambda, \alpha \rangle + \frac{\langle \alpha, \alpha \rangle}{2} \in 2\mathbb{Z} + 1$  for each  $\alpha \in \mathcal{S}$  that satisfies  $\chi_\mu(\sqrt{2}\alpha) = -1$ . So  $\lambda, \mu$  must satisfy  $\langle \lambda - \mu, \alpha \rangle \in 2\mathbb{Z}$ . That is,  $\chi_\lambda(\sqrt{2}\alpha_i) = \chi_\mu(\sqrt{2}\alpha_i)$  for any  $1 \leq i \leq d$  and

hence  $\chi_\lambda$  and  $\chi_\mu$  define the same character. Thus

$$\mathcal{U} \cdot W = \sum_{\alpha \in \mathcal{S}, \chi_\lambda(\sqrt{2}\alpha)=1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\lambda+\alpha},+} + \sum_{\alpha \in \mathcal{S}, \chi_\mu(\sqrt{2}\alpha)=-1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\lambda+\alpha},-}$$

is an irreducible  $\mathcal{U}$ -module. We denote this module by  $\widehat{(\lambda \ 0)}$ .

We now prove that  $\widehat{(\lambda + \beta \ 0)} = \widehat{(\lambda \ 0)}$  for any  $\beta \in L$ . It is clear that for any  $\beta \in 2L$ ,

$$V_{\frac{\lambda+\beta+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_{\lambda+\beta+\alpha}},+} = V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_{\lambda+\alpha}},+}$$

for  $\alpha \in \mathcal{S}$ . So  $\widehat{(\lambda + \beta \ 0)} = \widehat{(\lambda \ 0)}$ . If  $\beta$  does not lie in  $2L$ , we can assume  $\beta \in \mathcal{S}$  as  $\mathcal{S}$  is a coset representatives of  $2L$  in  $L$ . The result follows immediately from the definition of  $\mathcal{U} \cdot W$ .

Similarly, we can prove that for any  $\lambda, \mu \in L^\circ$ , when  $W = V_{\frac{\lambda}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\mu},-}$ ,

$$\mathcal{U} \cdot W = \sum_{\alpha \in \mathcal{S}, \chi_\lambda(\sqrt{2}\alpha)=1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_{\lambda+\alpha}},-} + \sum_{\alpha \in \mathcal{S}, \chi_\mu(\sqrt{2}\alpha)=-1} V_{\frac{\lambda+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_{\lambda+\alpha}},+}$$

is an irreducible  $\mathcal{U}$ -module which we denote by  $\widehat{(\lambda \ 1)}$ .

The number of inequivalent irreducible  $\mathcal{U}$ -modules of the form  $\widehat{(\lambda \ \epsilon)}$ ,  $\lambda \in \mathcal{T}$ ,  $\epsilon = 0, 1$  is  $2l$ .

Now we have in total  $\frac{l^2-l}{2} + 2l + 2l = \frac{l^2+7l}{2}$  irreducible  $\mathcal{U}$ -modules. Thus they are the inequivalent irreducible  $\mathcal{U}$ -modules.  $\square$

**Remark 4.5.** By Proposition 3.7 in [ADL], for any irreducible  $V_{\sqrt{2}L}$ -module  $V_{\lambda_i+\sqrt{2}L}$ , we have  $(V_{\lambda_i+\sqrt{2}L})' = V_{-\lambda_i+\sqrt{2}L}$ , where  $\lambda_i$  is any coset representative of  $\sqrt{2}L$  in  $(\sqrt{2}L)^\circ$ . By Proposition 3.3, it is clear that for any  $\lambda, \mu \in L^\circ$  with  $\lambda + L \neq \mu + L$ , and  $\epsilon = 0, 1$ , we have  $(\lambda \ \mu)' = (-\lambda \ -\mu)$ ,  $(\lambda \ \epsilon)' = (-\lambda \ \epsilon)$ , and  $(\lambda \ \epsilon)' = (-\lambda \ \epsilon)$ .

## 5 The quantum dimensions

Quantum dimensions have been systematically studied in [DXY, DRX]. It is proved that for a rational,  $C_2$ -cofinite, self-dual vertex operator algebra of CFT type, quantum dimensions of its irreducible modules have nice properties that are helpful in determining fusion products. The 2-permutation orbifold model  $(V_L \otimes V_L)^{\mathbb{Z}_2}$  we study here satisfies all the conditions and hence we can use quantum dimensions to determine some fusion rules. First we recall some notions and properties about quantum dimensions.

**Definition 5.1.** Let  $g$  be an automorphism of the vertex operator algebra  $V$  with order  $T$ . Let  $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M_{\lambda+n}$  be a  $g$ -twisted  $V$ -module, the formal character of  $M$  is defined as

$$\text{ch}_q M = \text{tr}_M q^{L(0)-c/24} = q^{\lambda-c/24} \sum_{n \in \frac{1}{T}\mathbb{Z}_+} (\dim M_{\lambda+n}) q^n,$$

where  $\lambda$  is the conformal weight of  $M$ .

We denote the holomorphic function  $\text{ch}_q M$  by  $Z_M(\tau)$ . Here and below,  $\tau$  is in the upper half plane  $\mathbb{H}$  and  $q = e^{2\pi i\tau}$ .

**Definition 5.2.** Let  $V$  be a vertex operator algebra and  $M$  a  $g$ -twisted  $V$ -module such that  $Z_V(\tau)$  and  $Z_M(\tau)$  exists. The quantum dimension of  $M$  over  $V$  is defined as

$$\text{qdim}_V M = \lim_{y \rightarrow 0} \frac{Z_M(iy)}{Z_V(iy)},$$

where  $y$  is real and positive.

Assume  $V$  is a rational,  $C_2$ -cofinite vertex operator algebra of CFT type with  $V \cong V'$ . Let  $M^0 \cong V, M^1, \dots, M^l$  be all inequivalent irreducible  $V$ -modules. Moreover, we assume the conformal weights  $\lambda_i$  of  $M^i$  are positive for all  $i > 0$ . Then we have the following properties of quantum dimensions [DJX]:

**Proposition 5.3.**  $\text{qdim}_V M^i \geq 1, \forall i = 0, \dots, l$ .

**Proposition 5.4.** For any  $i, j = 0, \dots, l$ ,

$$\text{qdim}_V (M^i \boxtimes M^j) = \text{qdim}_V M^i \cdot \text{qdim}_V M^j.$$

**Proposition 5.5.** A  $V$ -module  $M$  is a simple current if and only if  $\text{qdim}_V M = 1$ .

**Remark 5.6.** By Proposition 4.1 and [A4] we see that the vertex operator algebra  $(V_L \otimes V_L)^{\mathbb{Z}_2}$  satisfies all the assumptions required in [DJX] and thus we can apply these properties.

We obtain quantum dimensions of all irreducible  $(V_L \otimes V_L)^{\mathbb{Z}_2}$ -modules as follows:

**Proposition 5.7.** For  $\lambda, \mu \in L^\circ$  with  $\lambda + L \neq \mu + L, \epsilon = 0, 1$ , we have

$$\text{qdim}_{\mathcal{U}} (\widetilde{\lambda \epsilon}) = 1 \tag{5.1}$$

$$\text{qdim}_{\mathcal{U}} (\lambda \mu) = 2 \tag{5.2}$$

$$\text{qdim}_{\mathcal{U}} (\widehat{\lambda \epsilon}) = \sqrt{|L^\circ/L|} \tag{5.3}$$

*Proof.* Using the definition of quantum dimension we see that for any irreducible  $\mathcal{U}$ -module  $M$ ,

$$\text{qdim}_{\mathcal{U}} M = \frac{\text{qdim}_{\mathcal{V}} M}{\text{qdim}_{\mathcal{V}} \mathcal{U}}.$$

For any  $\alpha \in L^\circ, \alpha \in \mathcal{S}$ , by fusion rules of irreducible  $V_{\sqrt{2}L}$ - and  $V_{\sqrt{2}L}^+$ -modules in Proposition 3.2 and Proposition 3.5, we see that  $V_{\sqrt{2}\lambda + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^+$  is a simple current  $\mathcal{V}$ -module. By Proposition 5.4 and Proposition 5.5,  $V_{\sqrt{2}\lambda + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^+$  is of quantum dimension 1 as irreducible  $\mathcal{V}$ -module. Thus  $\text{qdim}_{\mathcal{V}} (\widetilde{\lambda \epsilon}) = 2^d$  and  $\text{qdim}_{\mathcal{V}} \mathcal{U} = 2^d$ . Thus we obtain  $\text{qdim}_{\mathcal{U}} (\widetilde{\lambda \epsilon}) = 1, \epsilon = 0, 1$ .

For  $\alpha \in \mathcal{S}$ ,  $\lambda, \mu \in L^\circ$  with  $\lambda + L \neq \mu + L$ , we have  $2 \left( \frac{\lambda - \mu + \alpha}{\sqrt{2}} \right) \notin \sqrt{2}L$ . Thus by fusion rules of irreducible  $V_{\sqrt{2}L}^+$ -modules in Proposition 3.5, quantum dimension of  $V_{\frac{\lambda - \mu + \alpha}{\sqrt{2}} + \sqrt{2}L}$  is 2 as irreducible  $V_{\sqrt{2}L}^+$ -module. By Proposition 5.4,

$$q \dim_{\mathcal{V}}(V_{\frac{\lambda + \mu + \alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\lambda - \mu + \alpha}{\sqrt{2}} + \sqrt{2}L}) = q \dim_{\sqrt{2}L} V_{\frac{\lambda + \mu + \alpha}{\sqrt{2}} + \sqrt{2}L} \cdot q \dim_{V_{\sqrt{2}L}^+} V_{\frac{\lambda + \mu + \alpha}{\sqrt{2}} + \sqrt{2}L} = 2.$$

Therefore we get  $q \dim_{\mathcal{V}}(\lambda \mu) = 2 \cdot 2^d = 2^{d+1}$  and hence we prove (5.2).

To prove (5.3), first we recall from [DJX] that  $\text{glob}(V) = \sum_M (q \dim M)^2$  where  $M$  runs over all irreducible modules of  $V$ . By Proposition 5.4,  $\text{glob}(V_L \otimes V_L) = (\text{glob}(V_L))^2$ . Now we have

$$\text{glob}(V_L \otimes V_L) = \left( \sum_{\lambda \in \mathcal{T}} (q \dim V_{\lambda+L})^2 \right)^2 = |\mathcal{T}|^2 = |L^\circ/L|^2 = l^2$$

as  $q \dim V_{\lambda+L} = 1$  for any irreducible  $V_L$ -module  $V_{\lambda+L}$ . Set  $q \dim_{\mathcal{U}}(\widehat{\lambda \epsilon}) = x$ , by quantum dimensions of irreducible  $\mathcal{U}$ -modules  $(\widehat{\lambda \epsilon})$  and  $(\lambda \mu)$  we obtain above, we have

$$\text{glob}(V^G) = \frac{l^2 - l}{2} \cdot 2^2 + 2l \cdot 1^2 + 2l \cdot x^2.$$

It is proved in [DRX] that  $\text{glob}(V^G) = |G|^2 \text{glob}(V)$ . Therefore we get  $2^2 \cdot \frac{l^2 - l}{2} + 2l \cdot 1^2 + 2l \cdot x^2 = 2^2 \cdot l^2$ . Solving the equation gives  $x = \sqrt{l}$  and thus  $q \dim_{\mathcal{U}}(\widehat{\lambda \epsilon}) = \sqrt{|L^\circ/L|}$ .  $\square$

## 6 Fusion Rules

In this section, we use the quantum dimensions obtained in the previous section and the fusion rules of irreducible  $V_{\sqrt{2}L}$ - and  $V_{\sqrt{2}L}^+$ -modules in [DL1, ADL] to determine the fusion products of the 2-permutation orbifold model.

**Theorem 6.1.** Let  $L$  be as before. Let  $\lambda, \mu, \gamma, \delta \in L^\circ$ ,  $\epsilon, \epsilon_1 = 0, 1$ .

(a) (i)

$$(\widehat{\lambda \epsilon}) \boxtimes (\widehat{\gamma \epsilon_1}) = (\lambda + \gamma \epsilon + \epsilon_1), \quad (6.1)$$

(ii) if  $\gamma + L \neq \delta + L$ , then

$$(\widehat{\lambda \epsilon}) \boxtimes (\gamma \delta) = (\lambda + \gamma \lambda + \delta), \quad (6.2)$$

(iii) if  $\lambda + L \neq \mu + L$ ,  $\gamma + L \neq \delta + L$ ,  $\lambda + \gamma + L = \mu + \delta + L$ , and  $\mu + \gamma + L = \lambda + \delta + L$ , then

$$(\lambda \mu) \boxtimes (\gamma \delta) = (\widehat{\lambda + \gamma 0}) + (\widehat{\lambda + \gamma 1}) + (\widehat{\mu + \gamma 0}) + (\widehat{\mu + \gamma 1}), \quad (6.3)$$

(iv) if  $\lambda + L \neq \mu + L$ ,  $\gamma + L \neq \delta + L$ ,  $\lambda + \gamma + L \neq \mu + \delta + L$ , and  $\mu + \gamma + L = \lambda + \delta + L$ , then

$$(\lambda \mu) \boxtimes (\gamma \delta) = (\lambda + \gamma \mu + \delta) + (\widehat{\mu + \gamma 0}) + (\widehat{\mu + \gamma 1}), \quad (6.4)$$

(v) if  $\lambda + L \neq \mu + L$ ,  $\gamma + L \neq \delta + L$ ,  $\lambda + \gamma + L \neq \mu + \delta + L$  and  $\mu + \gamma + L \neq \lambda + \delta + L$ , then

$$(\lambda \mu) \boxtimes (\gamma \delta) = (\lambda + \gamma \mu + \delta) + (\mu + \gamma \lambda + \delta). \quad (6.5)$$

(b) If  $\lambda + L \neq \mu + L$ ,

$$(\lambda \mu) \boxtimes (\gamma \epsilon) = (\lambda + \mu + \gamma 0) + (\lambda + \mu + \gamma 1), \epsilon = 0, 1, \quad (6.6)$$

$$(\widetilde{\lambda \epsilon}) \boxtimes (\widetilde{\mu \epsilon_1}) = (2\lambda + \mu \epsilon + \epsilon_1). \quad (6.7)$$

(c) Let  $G = \{\gamma \in \mathcal{T} | 2\gamma \in L\}$ ,

(i) if  $\frac{\lambda+\mu}{2} \in L^\circ$ ,

$$(\widetilde{\lambda \epsilon}) \boxtimes (\widetilde{\mu \epsilon_1}) = \sum_{\gamma \in G} \left( \frac{\lambda + \mu}{2} + \gamma \epsilon - \epsilon_1 \right) + \sum_{\delta \in L^\circ, \delta \neq \frac{\mu+\lambda}{2} + \gamma, \gamma \in G} (\lambda + \mu - \delta \delta), \quad (6.8)$$

(ii) if  $\frac{\lambda+\mu}{2} \notin L^\circ$ ,

$$(\widetilde{\lambda \epsilon}) \boxtimes (\widetilde{\mu \epsilon_1}) = \sum_{\delta \in L^\circ, \delta \neq \lambda + \mu - \delta} (\lambda + \mu - \delta \delta). \quad (6.9)$$

*Proof.* Consider fusion product of  $V_{\lambda+L} \otimes V_{\mu+L}$  and  $V_{\gamma+L} \otimes V_{\delta+L}$  as irreducible  $V_L \otimes V_L$ -modules. By fusion rules in Proposition 3.2 and Theorem 2.10 in [ADL], as irreducible  $V_L \otimes V_L$ -modules, we have the following fusion product:

$$(V_{\lambda+L} \otimes V_{\mu+L}) \boxtimes (V_{\gamma+L} \otimes V_{\delta+L}) = V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L}. \quad (6.10)$$

*Proof of (6.1):* If  $\lambda + L = \mu + L$  and  $\gamma + L = \delta + L$ , then  $\lambda + \gamma + L = \mu + \delta + L$ . Moreover,  $V_{\lambda+L} \otimes V_{\mu+L} \cong (\widetilde{\lambda 0}) + (\widetilde{\lambda 1})$ ,  $V_{\gamma+L} \otimes V_{\delta+L} \cong (\widetilde{\gamma 0}) + (\widetilde{\gamma 1})$ , and  $V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L} \cong (\widetilde{\lambda + \gamma 0}) + (\widetilde{\lambda + \gamma 1})$  as  $\mathcal{U}$ -modules.

First we consider fusion rule  $N_{\mathcal{U}} \left( \frac{(\widetilde{\lambda+\gamma \epsilon'})}{(\widetilde{\lambda 0}) (\widetilde{\gamma 0})} \right)$ ,  $\epsilon' \in \{0, 1\}$ . Take  $V = V_L \otimes V_L$  and  $U = \mathcal{U}$  in Proposition 2.9 in [ADL], then (6.10) implies

$$1 = N_{V_L \otimes V_L} \left( \frac{V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L}}{V_{\lambda+L} \otimes V_{\mu+L} \quad V_{\gamma+L} \otimes V_{\delta+L}} \right) \leq N_{\mathcal{U}} \left( \frac{(\widetilde{\lambda+\gamma 0}) + (\widetilde{\lambda+\gamma 1})}{(\widetilde{\lambda 0}) (\widetilde{\gamma 0})} \right).$$

So  $N_{\mathcal{U}} \left( \frac{(\widetilde{\lambda+\gamma \epsilon'})}{(\widetilde{\lambda 0}) (\widetilde{\gamma 0})} \right) = 0$  or  $1$ .

Now take  $V = \mathcal{U}$  and  $U = \mathcal{V}$  in Proposition 2.9 in [ADL], then

$$N_{\mathcal{U}} \left( \frac{(\widetilde{\lambda+\gamma \epsilon'})}{(\widetilde{\lambda 0}) (\widetilde{\gamma 0})} \right) \leq N_{\mathcal{V}} \left( \frac{(\widetilde{\lambda+\gamma \epsilon'})}{V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \quad V_{\sqrt{2}\gamma+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+} \right)$$



Since  $(\widetilde{\lambda + \gamma \epsilon'}) = \sum_{\alpha \in \mathcal{S}} V_{\sqrt{2}(\lambda+\gamma) + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^\pm$  where  $\pm$  depends on value of  $\epsilon'$ , we have

$$\begin{aligned} & N_{\mathcal{V}} \left( \begin{array}{c} (\widetilde{\lambda + \gamma \epsilon'}) \\ V_{\sqrt{2}\lambda + \sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \quad V_{\sqrt{2}\gamma + \sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \end{array} \right) \\ &= \sum_{\alpha \in \mathcal{S}} N_{\mathcal{V}} \left( \begin{array}{c} V_{\sqrt{2}(\lambda+\gamma) + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \otimes V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^\pm \\ V_{\sqrt{2}\lambda + \sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \quad V_{\sqrt{2}\gamma + \sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \end{array} \right) \\ &= \sum_{\alpha \in \mathcal{S}} N_{V_{\sqrt{2}L}} \left( \begin{array}{c} V_{\sqrt{2}(\lambda+\gamma) + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \\ V_{\sqrt{2}\lambda + \sqrt{2}L} \quad V_{\sqrt{2}\gamma + \sqrt{2}L} \end{array} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{array}{c} V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^\pm \\ V_{\sqrt{2}L}^+ \quad V_{\sqrt{2}L}^+ \end{array} \right). \end{aligned}$$

It is clear that

$$N_{V_{\sqrt{2}L}} \left( \begin{array}{c} V_{\sqrt{2}(\lambda+\gamma) + \frac{\alpha}{\sqrt{2}} + \sqrt{2}L} \\ V_{\sqrt{2}\lambda + \sqrt{2}L} \quad V_{\sqrt{2}\gamma + \sqrt{2}L} \end{array} \right) = N_{V_{\sqrt{2}L}^+} \left( \begin{array}{c} V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^+ \\ V_{\sqrt{2}L}^+ \quad V_{\sqrt{2}L}^+ \end{array} \right) = 1$$

if and only if  $\alpha = 0$ .  $N_{V_{\sqrt{2}L}^+} \left( \begin{array}{c} V_{\frac{\alpha}{\sqrt{2}} + \sqrt{2}L}^- \\ V_{\sqrt{2}L}^+ \quad V_{\sqrt{2}L}^+ \end{array} \right) = 0$  for  $\alpha \in \mathcal{S}$  forces  $\epsilon' = 0$ . So  $N_{\mathcal{U}} \left( \begin{array}{c} (\widetilde{\lambda + \gamma 0}) \\ (\widetilde{\lambda 0}) \quad (\widetilde{\gamma 0}) \end{array} \right) \leq 1$  and  $N_{\mathcal{U}} \left( \begin{array}{c} (\widetilde{\lambda + \gamma 1}) \\ (\widetilde{\lambda 0}) \quad (\widetilde{\gamma 0}) \end{array} \right) = 0$ . By quantum dimensions in Proposition 5.7, we get  $(\widetilde{\lambda 0}) \boxtimes (\widetilde{\gamma 0}) = (\widetilde{\lambda + \gamma 0})$ . We can similarly prove that for  $\epsilon, \epsilon_1 = 0, 1$ , we have

$$(\widetilde{\lambda \epsilon}) \boxtimes (\widetilde{\gamma \epsilon_1}) = (\widetilde{\lambda + \gamma \epsilon + \epsilon_1}).$$

Thus (6.1) has been proved.

*Proof of (6.2):* We now have  $V_{\lambda+L} \otimes V_{\lambda+L} \cong (\widetilde{\lambda 0}) + (\widetilde{\lambda 1})$ ,  $V_{\gamma+L} \otimes V_{\delta+L} \cong (\gamma \delta)$ , and  $V_{\lambda+\gamma+L} \otimes V_{\lambda+\delta+L} \cong (\lambda + \gamma \lambda + \delta)$  as irreducible  $\mathcal{U}$ -modules. Take  $V = V_L \otimes V_L$  and  $U = \mathcal{U}$  in Proposition 2.9 in [ADL], then (6.10) implies

$$1 = N_{V_L \otimes V_L} \left( \begin{array}{c} V_{\lambda+\gamma+L} \otimes V_{\lambda+\delta+L} \\ V_{\lambda+L} \otimes V_{\lambda+L} \quad V_{\gamma+L} \otimes V_{\delta+L} \end{array} \right) \leq N_{\mathcal{U}} \left( \begin{array}{c} (\widetilde{\lambda + \gamma \lambda + \delta}) \\ (\widetilde{\lambda \epsilon}) \quad (\gamma \delta) \end{array} \right)$$

By quantum dimensions in Proposition 5.7 and Proposition 5.4, we see that

$$q \dim_{\mathcal{U}} \left( (\widetilde{\lambda \epsilon}) \boxtimes (\gamma \delta) \right) = 2.$$

So  $N_{\mathcal{U}} \left( \begin{array}{c} (\widetilde{\lambda + \gamma \lambda + \delta}) \\ (\widetilde{\lambda \epsilon}) \quad (\gamma \delta) \end{array} \right) = 1$  and hence  $(\widetilde{\lambda \epsilon}) \boxtimes (\gamma \delta) = (\widetilde{\lambda + \gamma \mu + \delta})$ , as desired.

*Proof of (6.3):* We have  $V_{\lambda+L} \otimes V_{\mu+L} \cong (\lambda \mu)$ ,  $V_{\gamma+L} \otimes V_{\delta+L} \cong (\gamma \delta)$ ,  $V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L} \cong (\widetilde{\lambda + \gamma 0}) + (\widetilde{\lambda + \gamma 1})$ , and  $V_{\mu+\gamma+L} \otimes V_{\lambda+\delta+L} \cong (\mu + \gamma 0) + (\mu + \gamma 1)$  as  $\mathcal{U}$ -modules. By the fusion product in (6.10), we know there is a nonzero intertwining operator

$$I \in I_{V_L \otimes V_L} \left( \begin{array}{c} V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L} \\ V_{\lambda+L} \otimes V_{\mu+L} \quad V_{\gamma+L} \otimes V_{\delta+L} \end{array} \right).$$

Let  $P_\epsilon$  be the projection of  $V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L}$  to  $(\widetilde{\lambda+\gamma\epsilon})$  for  $\epsilon = 0, 1$ . Then  $P_\epsilon I$  is a nonzero intertwining operator in  $I_{\mathcal{U}} \left( \begin{smallmatrix} (\widetilde{\lambda+\gamma\epsilon}) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right)$ . So  $N_{\mathcal{U}} \left( \begin{smallmatrix} (\widetilde{\lambda+\gamma\epsilon}) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right) \geq 1$  for  $\epsilon = 0, 1$ . Since  $(\lambda\mu)$  and  $(\mu\lambda)$  are isomorphic  $\mathcal{U}$ -modules,  $N_{\mathcal{U}} \left( \begin{smallmatrix} (\widetilde{\mu+\gamma\epsilon}) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right) \geq 1$  for  $\epsilon = 0, 1$ .

By Proposition 5.4 and quantum dimensions in Proposition 5.7 we see that

$$q \dim_{\mathcal{U}} ((\lambda\mu) \boxtimes (\gamma\delta)) = 4.$$

So we obtain (6.3).

*Proof of (6.4):* Now we have  $V_{\lambda+L} \otimes V_{\mu+L} \cong (\lambda\mu)$ ,  $V_{\gamma+L} \otimes V_{\delta+L} \cong (\gamma\delta)$ ,  $V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L} \cong (\lambda+\gamma\mu+\delta)$ , and  $V_{\mu+\gamma+L} \otimes V_{\lambda+\delta+L} \cong (\mu+\gamma\lambda+\delta)$  as  $\mathcal{U}$ -modules. Take  $V = V_L \otimes V_L$  and  $U = \mathcal{U}$  in Proposition 2.9 in [ADL], then (6.10) implies

$$1 = N_{V_L \otimes V_L} \left( \begin{smallmatrix} V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L} \\ V_{\lambda+L} \otimes V_{\mu+L} \quad V_{\gamma+L} \otimes V_{\delta+L} \end{smallmatrix} \right) \leq N_{\mathcal{U}} \left( \begin{smallmatrix} (\lambda+\gamma\mu+\delta) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right).$$

So  $N_{\mathcal{U}} \left( \begin{smallmatrix} (\lambda+\gamma\mu+\delta) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right) \geq 1$ .

Using the condition  $\mu+\gamma+L = \lambda+\delta+L$  and the proof of (6.3) gives  $N_{\mathcal{U}} \left( \begin{smallmatrix} (\widetilde{\mu+\gamma\epsilon}) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right) \geq 1$  for  $\epsilon = 0, 1$ . Applying the formula  $q \dim_{\mathcal{U}} ((\lambda\mu) \boxtimes (\gamma\delta)) = 4$  again to obtain (6.4).

*Proof of (6.5):* Now we have  $V_{\lambda+L} \otimes V_{\mu+L} \cong (\lambda\mu)$ ,  $V_{\gamma+L} \otimes V_{\delta+L} \cong (\gamma\delta)$ ,  $V_{\lambda+\mu+L} \otimes V_{\mu+\delta+L} \cong (\lambda+\gamma\mu+\delta)$  and  $V_{\mu+\gamma+L} \otimes V_{\lambda+\delta+L} \cong (\mu+\gamma\lambda+\delta)$  as  $\mathcal{U}$ -modules. From the proof of (6.4) we see that

$$1 = N_{V_L \otimes V_L} \left( \begin{smallmatrix} V_{\lambda+\gamma+L} \otimes V_{\mu+\delta+L} \\ V_{\lambda+L} \otimes V_{\mu+L} \quad V_{\gamma+L} \otimes V_{\delta+L} \end{smallmatrix} \right) \leq N_{\mathcal{U}} \left( \begin{smallmatrix} (\lambda+\gamma\mu+\delta) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right)$$

and

$$1 = N_{V_L \otimes V_L} \left( \begin{smallmatrix} V_{\mu+\gamma+L} \otimes V_{\lambda+\delta+L} \\ V_{\mu+L} \otimes V_{\lambda+L} \quad V_{\gamma+L} \otimes V_{\delta+L} \end{smallmatrix} \right) \leq N_{\mathcal{U}} \left( \begin{smallmatrix} (\mu+\gamma\lambda+\delta) \\ (\mu\lambda) (\gamma\delta) \end{smallmatrix} \right).$$

Notice that as irreducible  $\mathcal{U}$ -modules,  $(\mu\lambda) \cong (\lambda\mu)$ . So  $N_{\mathcal{U}} \left( \begin{smallmatrix} (\mu+\gamma\lambda+\delta) \\ (\lambda\mu) (\gamma\delta) \end{smallmatrix} \right) \geq 1$ . The result follows immediately by the fact that  $q \dim_{\mathcal{U}} (\lambda+\gamma\mu+\delta) = q \dim_{\mathcal{U}} (\mu+\gamma\lambda+\delta) = 2$ .

*Proof of (6.6):* First by Proposition 5.4 and quantum dimensions in Proposition 5.7

$$q \dim_{\mathcal{U}} ((\lambda\mu) \boxtimes (\widetilde{\gamma\epsilon})) = q \dim_{\mathcal{U}} (\lambda\mu) \cdot q \dim_{\mathcal{U}} (\widetilde{\gamma\epsilon}) = 2\sqrt{|L^\circ/L|}.$$

By fusion rules in Proposition 3.5, for any  $\alpha, \beta \in \mathcal{S}$ ,  $N_{V_{\sqrt{2}L}^+} \left( \begin{smallmatrix} W \\ V_{\frac{\lambda-\mu+\alpha}{\sqrt{2}}+\sqrt{2}L} \quad V_{\sqrt{2}L}^{T_{\gamma+\beta,\pm}} \end{smallmatrix} \right) \neq 0$

only if  $W = V_{\sqrt{2}L}^{T_{\delta,\pm}}$  for some  $\delta \in L^\circ$ . So  $N_{\mathcal{U}} \left( \begin{smallmatrix} (\gamma_1\delta) \\ (\lambda\mu) (\widetilde{\gamma_0}) \end{smallmatrix} \right) = N_{\mathcal{U}} \left( \begin{smallmatrix} (\gamma_2\epsilon) \\ (\lambda\mu) (\widetilde{\gamma_0}) \end{smallmatrix} \right) = 0$  for any  $\gamma_1, \gamma_2, \delta \in L^\circ$  with  $\gamma_1+L \neq \delta+L$  and  $\epsilon = 0, 1$ . Using the classification of irreducible modules in Proposition 4.4 we see that  $(\lambda\mu) \boxtimes (\widetilde{\gamma\epsilon}) = (\widetilde{\delta_1\epsilon_1}) + (\widetilde{\delta_2\epsilon_2})$  for some  $\delta_1, \delta_2 \in L^\circ$  and  $\epsilon_1, \epsilon_2 \in \{0, 1\}$ . So we only need to determine  $\delta_1, \delta_1, \epsilon_1$  and  $\epsilon_2$ .

We first determine  $N_{\mathcal{U}} \left( \begin{smallmatrix} (\widehat{\delta} \ 0) \\ (\lambda \ \mu) \ (\widehat{\gamma \ 0}) \end{smallmatrix} \right)$  for  $\delta \in L^\circ$ . From Proposition 2.9 in [ADL] with  $V = \mathcal{U}$ ,  $U = \mathcal{V}$ , we have

$$N_{\mathcal{U}} \left( \begin{smallmatrix} (\widehat{\delta} \ 0) \\ (\lambda \ \mu) \ (\widehat{\lambda_1 \ 0}) \end{smallmatrix} \right) \leq N_{\mathcal{V}} \left( \begin{smallmatrix} (\widehat{\delta} \ 0) \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right).$$

From Proposition 4.4 we have

$$\begin{aligned} & N_{\mathcal{V}} \left( \begin{smallmatrix} (\widehat{\delta} \ 0) \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right) \\ & \leq \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=1} N_{\mathcal{V}} \left( \begin{smallmatrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{T_{\chi\delta+\alpha}}{\sqrt{2}L}} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right) \\ & + \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=-1} N_{\mathcal{V}} \left( \begin{smallmatrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{T_{\chi\delta+\alpha}}{\sqrt{2}L}} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right) \\ & = \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=1} N_{V_{\sqrt{2}L}} \left( \begin{smallmatrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \end{smallmatrix} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{smallmatrix} V_{\frac{T_{\chi\delta+\alpha}}{\sqrt{2}L}} \\ V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right) \\ & + \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=-1} N_{V_{\sqrt{2}L}} \left( \begin{smallmatrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \end{smallmatrix} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{smallmatrix} V_{\frac{T_{\chi\delta+\alpha}}{\sqrt{2}L}} \\ V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right). \end{aligned}$$

It is clear from the fusion rules for vertex operator algebra  $V_{\sqrt{2}L}$  (see Proposition 3.2) that if  $\delta + L \neq \lambda + \mu + \gamma + L$  then

$$N_{\mathcal{V}} \left( \begin{smallmatrix} (\widehat{\delta} \ 0) \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right) = 0.$$

We now assume that  $\delta = \lambda + \mu + \gamma$ . For  $\alpha \in \mathcal{S}$  with  $\chi_\delta(\sqrt{2}\alpha) = 1$ ,

$$N_{V_{\sqrt{2}L}} \left( \begin{smallmatrix} V_{\frac{\lambda+\mu+\gamma+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \end{smallmatrix} \right) = N_{V_{\sqrt{2}L}^+} \left( \begin{smallmatrix} V_{\frac{T_{\chi\lambda+\mu+\gamma+\alpha}}{\sqrt{2}L}} \\ V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \ V_{\frac{T_{\chi\gamma}}{\sqrt{2}L}} \end{smallmatrix} \right) = 1$$

only if  $\alpha \in 2L$  and  $\chi_{\lambda+\mu+\gamma+\alpha} = \chi_\gamma^{\left(\frac{\lambda-\mu}{\sqrt{2}}\right)}$ . Note that  $\alpha \in \mathcal{S}$  lies in  $2L$  if and only if  $\alpha = 0$ . Since

$$\chi_{\lambda+\mu+\gamma}(\sqrt{2}\alpha_i) = (-1)^{\frac{\langle \alpha_i, \alpha_i \rangle}{2} + \langle \lambda+\mu+\gamma, \alpha_i \rangle},$$

and

$$\chi_\gamma^{\left(\frac{\lambda-\mu}{\sqrt{2}}\right)}(\sqrt{2}\alpha_i) = (-1)^{\langle \lambda-\mu, \alpha_i \rangle} \chi_\gamma(\sqrt{2}\alpha_i) = (-1)^{\frac{\langle \alpha_i, \alpha_i \rangle}{2} + \langle \lambda-\mu+\gamma, \alpha_i \rangle},$$

we see that  $\chi_{\lambda+\mu+\gamma} = \chi_{\gamma}^{\left(\frac{\lambda-\mu}{\sqrt{2}}\right)}$ . So

$$\sum_{\alpha \in \mathcal{S}, \chi_{\delta}(\sqrt{2}\alpha)=1} N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{matrix} V_{\frac{T_{\chi_{\delta}+\alpha},+}{\sqrt{2}L}} \\ V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} V_{\frac{T_{\chi_{\gamma},+}}{\sqrt{2}L}} \right) = 1.$$

For  $\alpha \in \mathcal{S}$  with  $\chi_{\delta}(\sqrt{2}\alpha) = -1$ , we clearly have

$$N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \right) = 0$$

and consequently

$$\sum_{\alpha \in \mathcal{S}, \chi_{\delta}(\sqrt{2}\alpha)=-1} N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\frac{\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} V_{\frac{\gamma}{\sqrt{2}}+\sqrt{2}L} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{matrix} V_{\frac{T_{\chi_{\delta}+\alpha},-}{\sqrt{2}L}} \\ V_{\frac{\lambda-\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} V_{\frac{T_{\chi_{\gamma},+}}{\sqrt{2}L}} \right) = 0.$$

Thus

$$N_{\mathcal{U}} \left( \begin{matrix} (\lambda+\mu+\gamma \ 0) \\ (\lambda \ \mu) \ (\gamma \ 0) \end{matrix} \right) \leq 1 \text{ and } N_{\mathcal{U}} \left( \begin{matrix} (\delta \ 0) \\ (\lambda \ \mu) \ (\gamma \ 0) \end{matrix} \right) = 0 \text{ if } \delta + L \neq \lambda + \mu + \gamma + L.$$

Similarly,  $N_{\mathcal{U}} \left( \begin{matrix} (\lambda+\mu+\gamma \ 1) \\ (\lambda \ \mu) \ (\gamma \ 0) \end{matrix} \right) \leq 1$  and  $N_{\mathcal{U}} \left( \begin{matrix} (\delta \ 1) \\ (\lambda \ \mu) \ (\gamma \ 0) \end{matrix} \right) = 0$  if  $\delta + L \neq \lambda + \mu + \gamma + L$ . By quantum dimensions in Proposition 5.7, we get

$$N_{\mathcal{U}} \left( \begin{matrix} (\lambda+\mu+\gamma \ 0) \\ (\lambda \ \mu) \ (\gamma \ 0) \end{matrix} \right) = N_{\mathcal{U}} \left( \begin{matrix} (\lambda+\mu+\gamma \ 1) \\ (\lambda \ \mu) \ (\gamma \ 0) \end{matrix} \right) = 1,$$

as expected.

*Proof of (6.7):* First by Proposition 5.4 and quantum dimensions in Proposition 5.7

$$q \dim_{\mathcal{U}} \left( \widetilde{(\lambda \ \epsilon)} \boxtimes \widetilde{(\mu \ \epsilon_1)} \right) = q \dim_{\mathcal{U}} \widetilde{(\lambda \ \epsilon)} \cdot q \dim_{\mathcal{U}} \widetilde{(\mu \ \epsilon_1)} = \sqrt{|L^{\circ}/L|}.$$

By fusion rules in Proposition 3.5, for any  $\alpha, \beta \in \mathcal{S}$ ,  $N_{V_{\sqrt{2}L}^+} \left( \begin{matrix} W \\ V_{\frac{\alpha}{\sqrt{2}}+\sqrt{2}L}^+ \end{matrix} V_{\frac{T_{\mu+\beta},\pm}{\sqrt{2}L}} \right) \neq 0$  only if

$W = V_{\sqrt{2}L}^{T_{\gamma},\pm}$  for some  $\gamma \in L^{\circ}$ . So  $N_{\mathcal{U}} \left( \begin{matrix} (\gamma \ \delta) \\ (\lambda \ \epsilon) \ (\mu \ \epsilon_1) \end{matrix} \right) = N_{\mathcal{U}} \left( \begin{matrix} (\gamma_1 \ \epsilon) \\ (\lambda \ \epsilon) \ (\mu \ \epsilon_1) \end{matrix} \right) = 0$  for any  $\gamma, \delta, \gamma_1 \in L^{\circ}$  with  $\gamma + L \neq \delta + L$  and  $\epsilon, \epsilon_1 = 0, 1$ . As a result,  $\widetilde{(\lambda \ \epsilon)} \boxtimes \widetilde{(\mu \ \epsilon_1)} = \widetilde{(\delta \ \epsilon_2)}$  for some  $\delta \in L^{\circ}, \epsilon_1, \epsilon_2 \in \{0, 1\}$ .

First we consider  $N_{\mathcal{U}} \left( \begin{matrix} (\delta \ 0) \\ (\lambda \ 0) \ (\mu \ 0) \end{matrix} \right)$ . The proof of (6.6) shows that  $N_{\mathcal{U}} \left( \begin{matrix} (2\lambda+\mu \ 0) \\ (\lambda \ 0) \ (\mu \ 0) \end{matrix} \right) \leq 1$  and  $N_{\mathcal{U}} \left( \begin{matrix} (\delta \ 0) \\ (\lambda \ 0) \ (\mu \ 0) \end{matrix} \right) = 0$  if  $\delta + L \neq 2\lambda + \mu + L$ .

Now we consider  $N_{\mathcal{U}} \left( \begin{matrix} (\delta \ 1) \\ (\lambda \ 0) \ (\mu \ 0) \end{matrix} \right)$ . We have

$$N_{\mathcal{U}} \left( \begin{matrix} (\delta \ 1) \\ (\lambda \ 0) \ (\mu \ 0) \end{matrix} \right) \leq N_{\mathcal{V}} \left( \begin{matrix} (\delta \ 1) \\ V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \end{matrix} V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_{\mu},+}} \right).$$

Using  $\widehat{(\delta \ 1)} = \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=1} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\delta+\alpha},-} + \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=-1} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\delta+\alpha},+}$  gives

$$\begin{aligned}
& N_{\mathcal{V}} \left( \widehat{(\delta \ 1)} \right)_{V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\mu},+}} \\
& \leq \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=1} N_{\mathcal{V}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\delta+\alpha},-} \\ V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\mu},+} \end{matrix} \right) \\
& + \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=-1} N_{\mathcal{V}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\delta+\alpha},+} \\ V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\mu},+} \end{matrix} \right) \\
& = \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=1} N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\sqrt{2}\lambda+\sqrt{2}L} \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{matrix} V_{\sqrt{2}L}^{T_{\chi_\delta+\alpha},-} \\ V_{\sqrt{2}L}^+ \ V_{\sqrt{2}L}^{T_{\chi_\mu},+} \end{matrix} \right) \\
& + \sum_{\alpha \in \mathcal{S}, \chi_\delta(\sqrt{2}\alpha)=-1} N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\sqrt{2}\lambda+\sqrt{2}L} \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{matrix} V_{\sqrt{2}L}^{T_{\chi_\delta+\alpha},+} \\ V_{\sqrt{2}L}^+ \ V_{\sqrt{2}L}^{T_{\chi_\mu},+} \end{matrix} \right).
\end{aligned}$$

Clearly,  $N_{\mathcal{V}} \left( \widehat{(\delta \ 1)} \right)_{V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\mu},+}} = 0$  if  $\delta + L \neq 2\lambda + \mu + L$ . So we can assume that  $\delta = 2\mu + \lambda$ . Then for any  $\alpha \neq 0$  we have  $N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{\delta+\alpha}{\sqrt{2}}+\sqrt{2}L} \\ V_{\sqrt{2}\lambda+\sqrt{2}L} \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} \right) = 0$ . So

$$\begin{aligned}
& N_{\mathcal{V}} \left( \widehat{(2\lambda+\mu \ 1)} \right)_{V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\mu},+}} \\
& \leq N_{V_{\sqrt{2}L}} \left( \begin{matrix} V_{\frac{2\lambda+\mu}{\sqrt{2}}+\sqrt{2}L} \\ V_{\sqrt{2}\lambda+\sqrt{2}L} \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \end{matrix} \right) \cdot N_{V_{\sqrt{2}L}^+} \left( \begin{matrix} V_{\sqrt{2}L}^{T_{\chi_{2\lambda+\mu}},-} \\ V_{\sqrt{2}L}^+ \ V_{\sqrt{2}L}^{T_{\chi_\mu},+} \end{matrix} \right).
\end{aligned}$$

By fusion rules in Proposition 3.5,  $N_{V_{\sqrt{2}L}^+} \left( \begin{matrix} V_{\sqrt{2}L}^{T_{\chi_\delta+\alpha},-} \\ V_{\sqrt{2}L}^+ \ V_{\sqrt{2}L}^{T_{\chi_\mu},+} \end{matrix} \right) = 0$  for any  $\delta \in L^\circ$  and  $\alpha \in \mathcal{S}$ . Thus

$N_{\mathcal{V}} \left( \widehat{(\delta \ 1)} \right)_{V_{\sqrt{2}\lambda+\sqrt{2}L} \otimes V_{\sqrt{2}L}^+ \ V_{\frac{\mu}{\sqrt{2}}+\sqrt{2}L} \otimes V_{\sqrt{2}L}^{T_{\chi_\mu},+}} = 0$  for any  $\delta \in L^\circ$ . By counting quantum dimensions, we obtain

$$\widetilde{(\lambda \ 0)} \boxtimes \widehat{(\mu \ 0)} = \widehat{(2\lambda + \mu \ 0)}.$$

Use similar argument, we can prove for  $\epsilon, \epsilon_1 = 0, 1$ ,  $\widetilde{(\lambda \ \epsilon)} \boxtimes \widehat{(\mu \ \epsilon_1)} = \widehat{(2\lambda + \mu \ \epsilon + \epsilon_1)}$ .

*Proof of (6.8):* Recall that  $\mathcal{T}$  is a complete set of representatives of  $L$  in  $L^\circ$  and  $|\mathcal{T}| = l$ . If  $\frac{\lambda+\mu}{2} \in L^\circ$ , by Proposition 2.11, Remark 4.5, and fusion product (6.7), we see that  $N_{\mathcal{U}} \left( \begin{matrix} \widetilde{(\frac{\lambda+\mu}{2} \ \epsilon - \epsilon_1)} \\ \widehat{(\lambda \ \epsilon)} \ \widehat{(\mu \ \epsilon_1)} \end{matrix} \right) = 1$ . Note that  $\widehat{(\lambda + \alpha \ \epsilon)} \cong \widehat{(\lambda \ \epsilon)}$  for  $\lambda \in \mathcal{T}, \alpha \in L$  and  $\epsilon = 0, 1$ . Then by

Proposition 2.11, Remark 4.5, and fusion product (6.7), we also have  $N_{\mathcal{U}} \left( \frac{(\frac{\lambda+\mu}{2} + \gamma \epsilon - \epsilon_1)}{(\frac{\lambda+2\gamma}{2} \epsilon) (\mu \epsilon_1)} \right) = 1$  for all  $\gamma \in G$ . Clearly, for  $\gamma_1, \gamma_2 \in G$  with  $\gamma_1 \neq \gamma_2$ ,  $\frac{\lambda+\mu}{2} + \gamma_1 + L \neq \frac{\lambda+\mu}{2} + \gamma_2 + L$ . So  $(\frac{\lambda+\mu}{2} + \gamma_1 \epsilon - \epsilon_1)$  and  $(\frac{\lambda+\mu}{2} + \gamma_2 \epsilon - \epsilon_1)$  are not isomorphic  $\mathcal{U}$ -modules. Also notice that for any  $\gamma \in G$ ,  $(\lambda + 2\gamma \epsilon) \cong (\lambda \epsilon)$ . Assume  $|G| = n$ , then the number of isomorphism classes of  $(\frac{\lambda+\mu}{2} + \gamma \epsilon - \epsilon_1)$  such that  $N_{\mathcal{U}} \left( \frac{(\frac{\lambda+\mu}{2} + \gamma \epsilon - \epsilon_1)}{(\lambda \epsilon) (\mu \epsilon_1)} \right) = 1$  is equal to  $n$ .

By Proposition 2.11, Remark 4.5, and fusion product (6.6), we see that  $N_{\mathcal{U}} \left( \frac{(\lambda + \mu - \delta \delta)}{(\lambda \epsilon) (\mu \epsilon_1)} \right) = 1$  for any  $\delta \in \mathcal{T}$  such that  $\delta + L \neq \lambda + \mu - \delta + L$ . That is,  $N_{\mathcal{U}} \left( \frac{(\mu + \lambda - \delta \delta)}{(\lambda \epsilon) (\mu \epsilon_1)} \right) = 1$  for all  $\delta \in \mathcal{T}$  satisfying that  $\delta$  cannot be written of the form  $\frac{\lambda+\mu}{2} + \gamma$  with  $\gamma \in G$ . The number of such  $\delta \in \mathcal{T}$  is  $l - n$ . Also note that  $(\lambda \mu) \cong (\mu \lambda)$  as irreducible  $\mathcal{U}$ -modules for any  $\lambda, \mu \in \mathcal{T}$ . Thus the number of isomorphism classes of  $(\lambda + \mu - \delta \delta)$  such that  $N_{\mathcal{U}} \left( \frac{(\lambda + \mu - \delta \delta)}{(\lambda \epsilon) (\mu \epsilon_1)} \right) = 1$  is  $\frac{l-n}{2}$ . By counting quantum dimensions, we see that

$$(\widehat{\lambda \epsilon}) \boxtimes (\widehat{\mu \epsilon_1}) = \sum_{\gamma \in G} \left( \frac{\lambda + \mu}{2} + \gamma \epsilon - \epsilon_1 \right) + \sum_{\delta \in \mathcal{T}, \delta \neq \frac{\lambda+\mu}{2} + \gamma, \gamma \in G} (\lambda + \mu - \delta \delta).$$

*Proof of (6.9):* By Proposition 2.11, Remark 4.5, and fusion product (6.6), we see that

$$N_{\mathcal{U}} \left( \frac{(\lambda + \mu - \delta \delta)}{(\lambda \epsilon) (\mu \epsilon_1)} \right) = 1 \text{ for any } \delta \in \mathcal{T} \text{ such that } \delta + L \neq \lambda + \mu - \delta + L.$$

Since  $\frac{\lambda+\mu}{2} \notin L^\circ$ , we see that every  $\delta \in \mathcal{T}$  satisfy such condition. Thus the number of such  $\delta$  is equal to  $l$ . Notice that  $(\lambda \mu) \cong (\mu \lambda)$  as irreducible  $\mathcal{U}$ -modules for any  $\lambda, \mu \in \mathcal{T}$ . Thus the number of isomorphism classes of  $(\lambda + \mu - \delta \delta)$  such that  $N_{\mathcal{U}} \left( \frac{(\lambda + \mu - \delta \delta)}{(\lambda \epsilon) (\mu \epsilon_1)} \right) = 1$  is  $\frac{l}{2}$ . Now the quantum dimension of  $\sum_{\delta \in \mathcal{T}, \delta + L \neq \lambda + \mu - \delta + L} (\lambda + \mu - \delta \delta)$  is  $l$  and the proof of (6.9) is complete.  $\square$

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